Construction of the Entropy Solution of Hyperbolic Conservation Laws by a Geometrical Interpretation of the Conservation Principle

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In this paper we consider scalar hyperbolic equations in one space dimension of the type

$$u_t(x, t) + \frac{d}{dx} f(u; x) = h(u; x)$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}, t > 0, \tag{1}$$

where $f \in C^1$ and h continuous w.r.t. u, x. The initial condition is assumed to be piecewise continuous. We present a new method for constructing the entropy solution of (1) at a fixed time $t = T > 0$ in *one* time step based on transporting the initial values along characteristics. If the solution of (1) is smooth, we get the exact solution; in case of shocks the multivalued graph of the initial data is corrected by a geometrical averaging technique via the conservation principle. The method is also applicable to a scalar equation in which there is a mild coupling between the physical dimensions in the problem, for example,

$$u_t(x, y) + \frac{d}{dx} f(u; x, y) + \frac{d}{dy} f(u; x, y) = h(u; x, y). \tag{2}$$

By a change of variables, (2) can be reduced to a quasi one-dimensional problem. We conjecture that the advantage of computing the entropy solution at a fixed time in one time step cannot easily be carried over to systems. But we have some hints that this might be possible in case of scalar equations in two space dimensions with arbitrary fluxes f_1, f_2. The CPU time depends only on the total number of shocks which occur in the entropy solution up to time T; the accuracy of the computed shock position is of order at least 10^{-2}. Since our method is not based on a time discretisation, questions (and problems) concerning stability and convergence do not arise. © 1991 Academic Press, Inc.

1. Introduction

The subject of the paper is to introduce the *geometrical shock correction* method (GSC method) for calculating the entropy solution of scalar hyperbolic conserva-

tion laws with source term in one space dimension. We consider the Cauchy problem

$$u_t + \frac{d}{dx} f(u; x) = h(u; x)$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}, \, t > 0,$$

(1.1)

where $f \in C^1$ and h is continuous w.r.t. u, x. The initial condition u_0 is assumed to be piecewise continuous. Furthermore, the GSC method is also applicable to scalar two-dimensional problems whenever the fluxes f_1, f_2 permit a reduction to quasi one-dimensional problems via a change of the independent variables. With further mild restrictions on the fluxes f_1, f_2 and the source term h, stated in Section 2, problem (1.1) has a global unique weak solution in the sense of Kruzkov [12].

Hyperbolic equations describe wave motions and (1.1) may describe flow problems in which dissipation of energy occurs, for example, flow in porous media with source terms present. A concrete example is given by the Buckley–Leverett equation in Section 5 of this paper.

The GSC method is a very efficient numerical method for constructing the solution of (1.1) in one space dimension (resp. two space dimensions in case of $f_1 = f_2$) at a fixed time $t = T > 0$ in **one** time step. This procedure is based on the classical method of characteristics and a new algorithm for determination of the shock positions via a conservation principle. If the transport of the initial condition along characteristics at time T leads to a "multivalued relation" (in this case the method of characteristics fails to solve (1.1) for $t = T$), at leat one shock wave occurs. The following (x, u)-diagram (Fig. 1) shows the "multivalued relation" (represented by a curve) of Example 5 in Section 5 at time $t = 0.4$. In Sections 3 and 4 we show that such ambiguities can be easily resolved by our GSC-method.
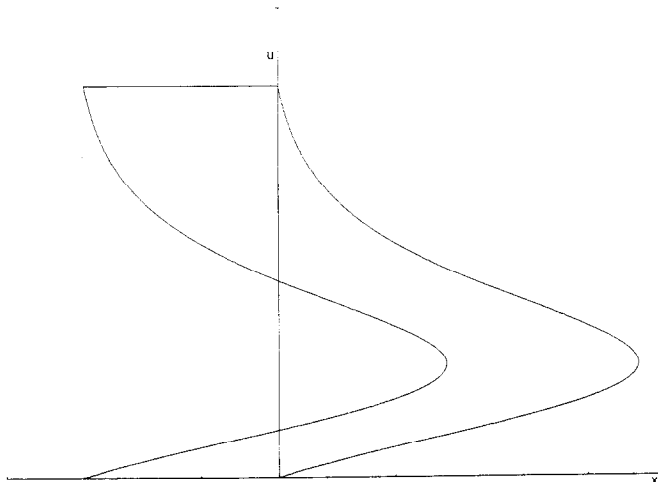

FIGURE 1

In 1972 Marle [13] described a heuristical method, based on physical observations, for constructing the weak solution from a three-valued relation of a two-phase flow problem in a porous medium. This method, where two areas must have the same measure, dates back to Brinkman's work [3] in 1949. He presumes that it might be possible to construct the solution consisting of a single shock by this physically motivated principle. A similar statement holds for the simplified "Tangential-Method," developed by Welge [15] in 1952, for constructing the approximate shock position of some special problem geometrically. The shock determination in [18] is described entirely on the fixed curve of the flux by drawing all chords with the equal area property in the (x, f)-plane (method of convex hull). In this approach, the shock position as well as the two limit states at time $t = T$ are given *implicitly* by solving an integral equation coupled with two nonlinear equations. In contrast, the GSC method is much simpler.

In 1981 Brenier [2] took up the concept of "conservation of areas" for his transport-collapse-method. This method, based also on the theory of characteristics and a certain averaging of the multivalued relation, leads to a convergent and stable *time discrete* numerical procedure.

Since the GSC method computes the exact solution in *one* time step, indeed the shock positions are at least correct to 10^{-2}, there arise no questions concerning stability, convergence, and the quality of the solution in the presence of shock and rarefaction waves. Hence, in contrast to any finite difference method, we have no time step restrictions by a CFL condition. The effect of the CFL condition is demonstrated by the following simple example in one space dimension, taken from [11]. Let

$$f(u) = \frac{u^2}{2}, \qquad u_0(x) = \begin{cases} 24; & x < 0.5 \\ 24 + \sin(\pi(x - 0.5)); & 0.5 < x < 2.5 \\ 24; & x > 2.5. \end{cases}$$

Choosing $\Delta x = 0.04$ results in $\Delta t \leqslant \Delta x / \max |f'(u)| \leqslant 1/625$. Selecting $\Delta t = 0.008$ and computing u at time $T = 2$ in the interval $[48, 51]$ with the TVD scheme of Harten, see [9], took around 31 min on an IBM AT 80286. This numerical solution is very smeared in the vicinity of the single shock appearing at $x_s = 49.544$.

In comparison, the GSC method gives the solution with excellent quality in 6.37 s, which is approximately 300 times faster.

The CPU time of the GSC method at time T is directly proportional to the number of appearing shocks; hence there might be only a small increase in the computing time for growing T, which again is totally in contrast to any finite difference method. In conclusion, the GSC method computes the exact solution for all generic situations especially for larger T in a fraction of time required for any finite difference method. However, it seems that there is no obvious extension of the GSC method to hyperbolic systems.

The paper is organized as follows. In Section 2 we pose the problem and give a brief summary of the method of characteristics before introducing the conservation

principle which leads to the geometrical shock correction in case of $u_t + (f(u))_x = 0$. In Section 3 we continue with the GSC method showing that the algorithm resolves any multivalued configuration and extend the method to the general equation (1.1) in one space dimension. Applications of GSC to special Cauchy problems in two space dimensions are given in Section 4. Finally, in Section 5, we show some numerical examples.

2. THE METHOD OF CHARACTERISTICS AND THE CONSERVATION PRINCIPLE

Consider the Cauchy problem for a single conservation law with source term h:

$$u_t + [f(u)]_x = h(u, x, t), \qquad x \in \mathbb{R}, \, t > 0$$
$$u(x, 0) = u_0(x). \tag{2.1}$$

We assume that the flux f, the source term h, and u_0 satisfy the following conditions (see [12]):

(i) $f, h, f_u, f_{ux}, f_x, h_u, h_x$ are continuous.

(ii) For $(u, x, t) \in [-M, M] \times \mathbb{R} \times [0, T]$, $M, T > 0$, the functions f_u and $g \equiv h - f_x$ are bounded,

(iii) $\sup\{h(0, x, t); \, (x, t) \in \mathbb{R} \times [0, T]\} \leqslant c_0 = \text{const}$, $\sup\{h_u(u, x, t); \, (x, t) \in \mathbb{R} \times [0, T]; \, -\infty < u < \infty\} \leqslant c_1 = \text{const}$,

(iv) u_0 is a bounded measurable function.

Remark. If f depends only on u and if $h \equiv 0$, we require only $f \in C^1$ and (iv). Condition (iii) prevents the escape of u from every compact subset of the state space. Since global smooth solutions do not exist in general, we consider weak (distributional) solutions, which are not unique. Therefore one needs an *entropy condition* in order to select the physically relevant solution, see, e.g., [12]. The permissible curves of discontinuity are related by the *Rankine–Hugoniot condition*,

$$s[u] = [f(u, x, t)],$$

where $[u] = u_l - u_r$ is the jump across a smooth curve Γ and $s = dx/dt$ is the speed of discontinuity.

Under the assumptions (i)–(iv) Kruzkov shows in [12] the existence, uniqueness, and stability w.r.t. the initial condition of the weak solution of (2.1).

After these preparations we first describe our method for the problem

$$u_t + [f(u)]_x = 0, \qquad x \in \mathbb{R}, \, t > 0,$$
$$u(x, 0) = u_0(x) \tag{2.2}$$

where either (a) u_0 is continuous or (b) u_0 is bounded with a finite number of jump discontinuities. Introducing the system of ordinary differential equations for (2.2)

$$\frac{du}{dt} = 0, \qquad \frac{dx}{dt} = f'(u) \tag{2.3}$$

and defining for each $(x, w) \in \mathbb{R}^2$,

$$(X(x, w; t), U(x, w; t)) = F(x, w; t)$$

to be the unique solution to (2.2) with (x, w) as initial value at $t = 0$, the following result is classical

PROPOSITION (Method of characteristics). *Let u_0 be continuous. Any classical (i.e., C^1) solution to (2.2) satisfies $G(t) = FG(0)$, where $G(t)$ denotes the graph to (2.3) at time t, $G(t) = \{(x, w) \in \mathbb{R}^2; w = u(x, t)\}$.*

In other words, the graph of the solution is transported by the flow F associated with (2.3). Of course, the method of characteristics fails to solve (2.2) in the large, no matter how smooth the initial data are. Hence, there is no reason for $FG(0)$ to remain a graph for every $t > 0$. Generally, $FG(0)$ becomes multivalued and defines no longer a solution of (2.2). Instead of $G(t)$ we consider in our method the *subgraph* $SG(t)$ of the solution at time t, given by the transport along characteristics,

$$SG(t) = \{(x, w) \in \mathbb{R}^2; w \leqslant u(x, t)\}.$$

As long as u remains smooth, we have $SG(t) = FSG(0)$. Afterwards $FSG(0)$ and $FG(0)$ define the same multivalued relation, see [2]. An illustration is given in Fig. 2. Explicit integration of (2.3) gives $F(x, w; t) = (x + tf'(w), w)$; hence
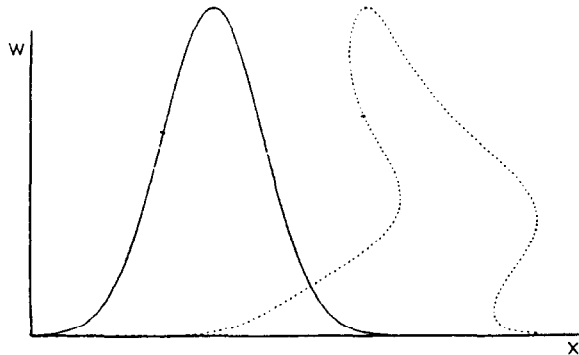
$$FG(0) = \{(x, w); (x - tf'(w), w) \in G(0)\}.$$

FIGURE 2

In the same way, $FSG(0) = \{(x, w); (x - tf'(w), w) \in SG(0)\}$. If u_0 is only piecewise continuous (as in the case of Riemann problems) the transport at time t does not give a connected curve. In the jump points of u_0 we insert the vertical line between the left and right limit values, i.e., for

$$u_0(x) = \begin{cases} u_1(x); & x < x_1 \\ u_2(x); & x_1 < x < x_2 \\ \vdots & \vdots \\ u_n(x); & x > x_n \end{cases}$$

we consider the modified initial condition

$$u_0^*(x) = \begin{cases} w = u_1(x); & x < x_1 \\ \{w; u_1(x_1) \leqslant w \leqslant u_2(x_1)\}; & u_1(x_1) < u_2(x_1) \text{ or} \\ \{w; u_2(x_1) \leqslant w \leqslant u_1(x_1)\}; & u_1(x_1) > u_2(x_1) \\ w = u_2(x); & x_1 < x < x_2 \\ \vdots & \vdots \\ w = u_n(x); & x > x_n. \end{cases}$$

Then the transport gives a connected curve. We shall motivate this modification by a simple Riemann problem for (2.2):

$$u_0(x) = \begin{cases} u_l; x < 0 \\ u_r; x > 0, u_l, u_r \in \mathbb{R}. \end{cases} \tag{2.4}$$

For simplicity we assume that $f \in C^2$, $f'' > 0$ and let $t = T > 0$ be arbitrary, but fixed. In the first case, $u_l < u_r$, the solution consists of a rarefaction wave which connects the two constant states u_l and u_r. Within the fan of this wave the solution is implicitly given by

$$x = Tf'(s), \qquad u_l \leqslant s \leqslant u_r.$$

The transport of (2.4) leaves a gap between $x = Tf'(u_l)$ and $x = Tf'(u_r)$, where the solution is a priori undefined, whereas the entropy solution is given by the transport of the *modified* initial condition.

In the case $u_l > u_r$, the two constant states are connected by a shock wave with speed $s = [f]/[u]$. The *limit characteristics* with slope $1/f'(u_l)$ (resp. $1/f'(u_r)$) define the multivalued region in the (x, t)-plane. The transport of the initial data at time $T > 0$ results in a disconnected multivalued relation. Inserting however $x = Tf'(s)$, $u_r \leqslant s \leqslant u_l$ (which is the transported vertical line $u_r \leqslant w \leqslant u_l$, inserted in the initial condition, see u_0^* above with $n = 2$ and $x_1 = 0$), the corresponding curve in the (x, u)-diagram results in a connected multivalued relation, see Fig. 3. B. van Leer [14] associates this case with the occurrence of an overturned centered compression wave.
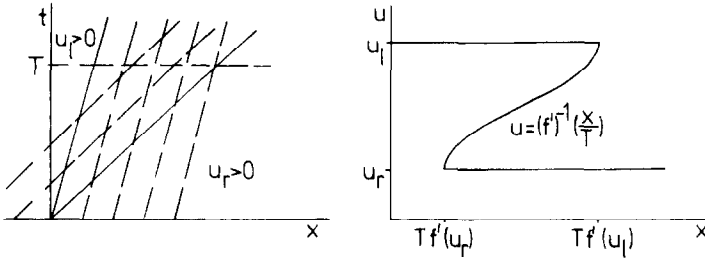
In generic situations one of the following two alternatives occur: Transport of the eventually modified initial data at time $T > 0$ results in either

(a)   a (unique) connected graph in the (x, u) space, which represents the entropy solution, or

(b)   in a multivalued relation in certain parts of the (x, u) space.

Case (b) indicates the presence of at least one shock wave. The following definition is useful in describing the method.

DEFINITION.   (a) An interval $[a, b] \subset \mathbb{R}$ is said to be an *isolated single-valued region* (ISR), if the mapping $H_T: x \rightarrow u(x, T)$, $x \in M$, u given by the transport, is unique for all $x \in [a, b]$.

(b)   A *multivalued region* (MR) is an interval which does not represent an isolated single valued region.

(c)   An interval $[x_l, x_r] \subset \mathbb{R}$ consisting of the union of multivalued regions, $\bigcup_{i=1}^{n} [x_{l_i}, x_{r_i}]$ is said to be an *isolated multivalued region* (IMR), if the following three conditions hold:

(i)   there exists no $M \subset [x_l, x_r]$, such that $H_T(x)$, $x \in M$, is unique for $x \in M$

(ii)   there exists $\varepsilon_1, \varepsilon_2 > 0$ such that H_T is unique in $\{x; \varepsilon_1 < x \leqslant x_l; x_r < x \leqslant \varepsilon_2\}$

(iii) $x_l = \min\{x_{l_i}; 1 \leqslant i \leqslant n\}$, $x_r = \max\{x_{r_i}; 1 \leqslant i \leqslant n\}$.

This definition enables us to divide the MR is a disjoint union of ISR and IMR.

Note.   The correction of the MR can be done separately for every IMR because of the uniqueness of the solution in the ISR.

3. THE GSC METHOD

We will show how to construct the entropy solution from the MR and that this can be done numerically in an efficient way. The transport combined with this

correction procedure is called the *GSC method*. The correction procedure is based on the following conservation principle. Let $[a, b]$ denote an IMR and $A(T)$ the (algebraic) area of the entropy solution at time $t = T > 0$ over $[a, b]$. The transport may give a multivalued relation $\hat{u}(x, T)$ and $C(T)$ represents the (algebraic) area of $\{(x, \hat{u}); a \leqslant x \leqslant b\}$; i.e., $C(T)$ is the area of the subgraph of u_0 given by the transport at time T. Then

$$A(T) = C(T)$$

defines the entropy solution uniquely. This is equivalent to requiring

$$\operatorname*{meas}_{x \in [a,b]} (SG(T) \backslash FSG(0)) = \operatorname*{meas}_{x \in [a,b]} (FSG(0) \backslash SG(T)), \tag{3.1}$$

where meas(A) is the Lebesgue measure of A, see [2]. This principle can be expressed in the following way: Assume that the multivalued relation over $[a, b]$ can be resolved by a single shock at position x_s. Then, the sum of the (algebraic) areas bounded by $\hat{u}(x, T)$, $x \leqslant x_s$, and the vertical line $x = x_s$ equals the sum of the areas between $x = x_s$ and $\hat{u}(x, T)$ for $u_0 \in C(\mathbb{R})$ this is shown by the example in Fig. 4.

The curve is split up in three parts, where $v_i(x)$, $i = 1, 2, 3$, denotes the arcs between the corresponding points P_a, P_1, P_2, and P_b, respectively. x_s is the unknown shock position. Then,

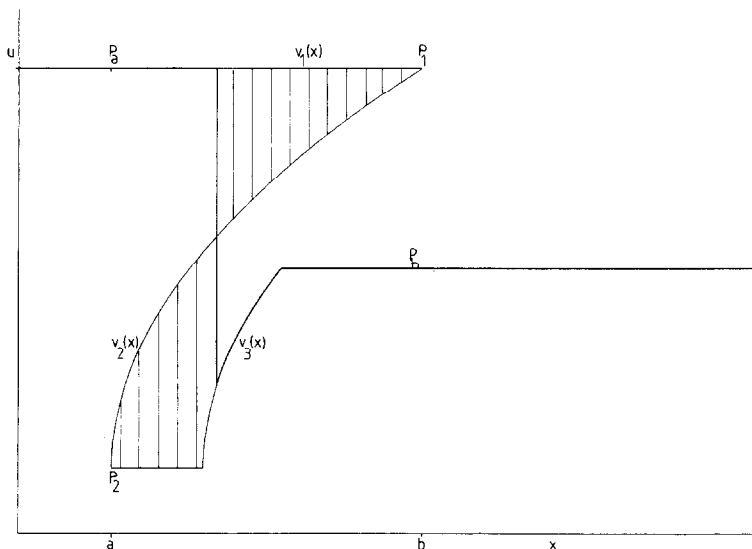$$C(T) = \int_a^b [v_1(x) - v_2(x) + v_3(x)] \, dx$$

FIGURE 4

and

$$A(T) = \int_a^{x_s} v_1(x)\,dx + \int_{x_s}^b v_3(x)\,dx,$$

since the shock connects $v_1(x_s-)$ and $v_3(x_s+)$. Hence,

$$C(T) - A(T) = \int_a^{x_s} [v_3(x) - v_2(x)]\,dx + \int_{x_s}^b [v_1(x) - v_2(x)]\,dx$$

or, equivalently,

$$\int_a^{x_s} [v_2(x) - v_3(x)]\,dx = \int_{x_s}^b [v_1(x) - v_2(x)]\,dx.$$

This determines x_s uniquely. The hatched areas in Fig. 4 have equal measure.

In order to resolve more complicated situations in an IMR we need a further definition.

DEFINITION.   A point $(x, \hat{u}(x, T))$ is said to be a *turning point* of $\hat{u}(x, T)$, if there exists a $\delta > 0$ such that for all points $(y, \hat{u}(y, T))$, $(z, \hat{u}(z, T)) \in K_\mu((x, \hat{u}(x, T)))$,

$$(z - x)(x - y) \leqslant 0$$

holds, where $K_\mu((x, \hat{u}(x, T)))$ denotes a circle with center $(x, \hat{u}(x, T))$ and radius μ, $0 < \mu \leqslant \delta$.

It is important to note that the configuration in IMR as well as the number and the size of such regions depend on u_0, f, and $t = T$. Furthermore, the conservation principle must be applied to every IMR separately. Only two situations have to be resolved in an IMR: (a) There are no interactions between the occurring waves, or (b) some (or all) waves have been interacted.

We remark that the transport of the (eventually) modified initial data at time T defines uniquely the configuration. Hence, we show that the GSC method always gives the solution in case of generic situations for the Cauchy problem (2.2).

The proof is by induction over the number of elementary waves, i.e., smooth states, rarefaction waves, and shocks, where the former two are represented by pieces of the curve in the IMR at time $T > 0$. The case of a single shock has been treated above. If only two waves of type shock/rarefaction resolve the ambiguities in the isolated region, there exist only two possibilities, either:

(i)   two shock waves, i.e., four turning points; or

(ii)   a shock wave and a rarefaction wave appear; now two turning points define the IMR.

Passing through the curve, two turning points are assigned to each single ambiguity. Possibility (i) has to be resolved as follows:

The provisional positions of the two shocks, x_{s_1}, x_{s_2}, are determined by principle (3.1). Depending of the values of x_{s_1} and x_{s_2}, one of the following three cases occur:

(a) If $x_{s_2} > x_{s_1}$, then there is no interaction between the shocks and they are separated by a smooth state. This resolution is unique.

(b) For $x_{s_1} > x_{s_2}$ remains a MR; thus the shocks have interacted. Again, the correct position x_s, $x_{s_2} < x_s < x_{s_1}$ of the single shock is given by (3.1).

(c) $x_{s_1} = x_{s_2}$, the resulting shock position has been found.

In case (ii) a single shock wave appears. In contrast to case (i) where, in addition to the geometrical correction procedure, the provisional positions of the shocks have to be compared, the shock position is given by our principle (3.1) automatically. The resulting shock may "devour" a part or all of the rarefaction wave.

Continuation of this process shows that every configuration of an IMR at time T can be split up uniquely is a connected single-valued relation via the conservation principle. This function represents the entropy solution over the considered interval.

In order to maintain accuracy in smooth regions at a final time T it is necessary to discretise the (eventually) modified initial condition sufficiently fine. This can be checked a priori, since the following inequality provides an upper bound on the distance between two neightbouring values of the graph.

Let $h = x_1 - x_2$ and $\hat{x}$ be a jump point of u_0. Then,

$$| (x_1 + Tf'(u_0(x_1))) - (x_2 + Tf'(u_0(x_2))) |$$

$$\leqslant h + T \, |f''(\eta)| \, |u_0(x_1) - u_0(x_2)|$$

$$\leqslant h + T \, |f''(\eta)| \, [\, |u_0(x_1) - u_0(\hat{x})| + |u_0(\hat{x}) - u_0(x_2)| \,]$$

$$\leqslant h[1 + T \, |f''(\eta)| \max[\, \max_{x \in [x_1, \hat{x}]} |u_0'(x)|; \, \max_{x \in [\hat{x}, x_2]} |u_0'(x)| \,]],$$

where $\eta \in [\min\{u_0(x_1), u_0(\hat{x}), u_0(x_2)\}, \max\{u_0(x_1), u_0(\hat{x}), u_0(x_2)\}]$.

Thus, linear interpolation is sufficient for representing the solution in smooth regions. Furthermore, a local refinement of the x-discretisation in smooth regions has no essential effect on the CPU time (in contrast to finite difference schemes).

Implementation of the algorithm:

(1) Discretise the (eventually modified) initial condition.

(2) Apply the method of characteristics, i.e., the transport of u_0 along characteristics at the prescribed time. In order to get a connected curve one has to interpolate (for an appropriate fine discretisation linear interpolation is sufficient) between the u-values.

(3) If $FSG(0)$ becomes multivalued, determine the first two turning points (discretisation of the initial condition leads to an oriented set of points). The provisional shock position is given by our principle (3.1) iteratively. Areas are computed by a quadrature rule; the summed trapezoidal rule is used in our numerical examples.

(4) Next one has to check whether interactions with possible rarefaction waves took place. If this happens, calculate the position of the resulting shock and continue with the next pair of turning points. Otherwise consider the next pair of turning points and determine the provisional shock position within the MR.

(5) Again check whether interactions between this second provisional shock with rarefactions take place (there can be m rarefaction waves between both pairs of turning points). If no interaction takes place, continue with the next pair of turning points. Otherwise find out whether the shock has interacted with the first shock or some of the rarefaction waves.

This procedure finishes with the resolution of the ambiguity, restricted by the last two turning points. Since it is possible that the first and the last shock interact, at the present all successive determined shock positions have to be considered as provisional. In the case of Riemann initial data one computes the (provisional) shock positions by the Rankine–Hugoniot condition. This is much faster than applying our conservation principle.

There is an interesting connection between the time-dependent version of the method of convex hull (applied to the flux) and the GSC method for (2.2) with Riemann initial data. In [5] Chang constructs qualitatively the piecewise smooth solution to (2.2) for n constant initial values under the assumption that $f \in C^2$ has at most finitely many inflection points. In his geometrical construction process the involved waves as well as their variations in u can be easily read off. However, this is not the case for the exact positions of the waves, especially after interactions. In the same way Whitham's method in [18] can be related to the time-depended version of the method of convex hull.

Extending our method to the general equation

$$u_t + \frac{d}{dx} f(u; x, t) = h(u; x, t),$$

we introduce the system of differential equations

$$\frac{du}{dt} = g(u; x, t), \qquad \frac{dx}{dt} = f_u(u; x, t), \qquad (3.2)$$

where $g \equiv h - f_x$. Now we have to transport u_0 (resp. u_0^*) by the flux associated with (3.2). In a couple of cases this can be done explicitly. It is well known, see [16], that system (3.2) need not have a global weak solution.

Equation (2.2) represents a special case of

$$[G(u)]_t + [F(u)]_x = 0, \qquad u(x, 0) = u_0(x). \qquad (3.3)$$

Assuming $F, G \in C^2$, $G' > 0$, and $H'(u) := (F'(u)/G'(u))'$ has at most finitely many inflection points, Ballou [1] shows existence and uniqueness of the weak solution.

Let $v = G(u)$. Since $G' > 0$, $u = G^{-1}(v)$ is well defined and with $A(v) \equiv F(G^{-1}(v))$, (3.3) reduces to

$$v_t + [A(v)]_x = 0, \qquad v_0(x) = G(u_0(x)).$$

This is equivalent to writing (3.3) along the characteristics

$$\frac{dG}{dt} = 0, \qquad \frac{dx}{dt} = H(u).$$

The transport is then given by $(x + tH(u), G(u))$. For the inversion of G at shock points select the appropriate left/right limit values.

4. APPLICATION OF THE GSC METHOD TO CONSERVATION LAWS IN TWO SPACE DIMENSIONS

In this section we consider the application of the GSC method to scalar conservation laws with a source term in two space dimensions

$$u_t + \frac{d}{dx} f_1(u; x, y) + \frac{d}{dy} f_2(u; x, y) + s(u; x, y) = 0, \qquad x, y \in \mathbb{R}, t > 0. \quad (4.1)$$

Let $f_1, f_2 \in C^1$, $s \in C$ w.r.t. u, x, y. Unfortunately, an extended GSC method is in general not directly applicable to (4.1). Of course, the method of characteristics gives the solution in regions where u is smooth, but it is a priori not clear how to construct the solution from the multivalued relation in some region. Inspired by the "equal area" principle one might think that in the $(x, y; u)$ space an "equal volume" approach would be suitable for determing shock surfaces. Indeed, this seems possible in the case of scalar two-dimensional conservation laws.

If, however, $f_1 = f_2 \equiv f$ in (4.1), which is the case in many applications, one can transform (4.1) in a quasi-one-dimensional problem which depends on a parameter. The application of the GSC method for every (fixed) parameter is then straightforward. Let $f_1 = f_2 \equiv f$. Introducing the coordinate transformation $(x, y) \to (\xi, \eta)$ defined by

$$\xi = \frac{x + y}{2}, \qquad \eta = \frac{x - y}{2}$$

and setting $u(x, y; t) = u(\xi, \eta; t)$, (4.1) reads

$$u_t + f_u(u; \xi + \eta, \xi - \eta) u_\xi = g(u; \xi + \eta, \xi - \eta) \quad (4.2)$$

with $g = -(s + f_\xi)$. Equation (4.2) represents, for each fixed parameter $\eta \in \mathbb{R}$, a one-dimensional problem. Hence, the solution along every plane $\{(\xi, \eta_1) \times t, \eta_1 = \text{const}\}$, can be determined independently from other planes $\{(\xi, \eta_2) \times t\}$. GSC can also be applied to (4.1) for slightly different situations.

LEMMA [17]. *Let $f_i = f_i(u)$, $f_i \in C^2$, $i = 1, 2$. Then, (4.1) can be transformed into a quasi-one-dimensional conservation law with source term by a linear transformation in the independent variables*

$$(x, y; t) \rightarrow (\xi(x, y; t), \eta(x, y; t), t)$$

if and only if

$$(f_1''/f_2'')' = 0.$$

Since (4.2) depends explicitly only on ξ, the discretisation in ξ should be much finer than in η. Furthermore, the computing time of the GSC method is proportional to the number of η values. Therefore we recommend a coarse discretisation in η and (linear) interpolation of the solution between neighboring η values.

An estimation in the order of magnitude of the computing effort is given by the number of discretisation points in the considered η-interval times the CPU time for computing the solution of the one-dimensional problem to the corresponding fixed η-value.

In [17] the second author shows that in case of $f_1 = f_2 \equiv f$, f convex, the Riemann problem can be solved directly with the GSC method without transforming the problem into a quasi-one-dimensional one. Here, the Cauchy problem

$$u_t + [f(u)]_x + [f(u)]_y = 0, \qquad u(x, y; 0) = u_0(x, y) \qquad (4.3)$$

with initial data piecewise constant on a finite number of wedges focused in a single point in the (x, y) plane is defined as a two-dimensional Riemann problem; w.l.o.g. this point can be taken to be $x = 0$, $y = 0$.

Since the solution of (4.3) is invariant under dilations $(x, y; t) \rightarrow (cx, cy; ct)$, $c > 0$, we may describe a solution completely by describing it along the plane $t = 1$. In the region $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 \geqslant r_1^2, r_1 > 0\}$, where r_1 depends only on f and u_0 along the considered plane $t = t_1$, the solution is obtained as the solution of non-interacting one-dimensional Riemann problems, due to the principle of finite domain of dependence. Such one-dimensional problems can be solved directly with the GSC method as well as certain interactions, which occur by extrapolating the one-dimensional solution into the region $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 < r_1^2\}$ by "fitting together" the nonlinear waves. Some interactions like shock/rarefaction, can be resolved by determination of the position of the resulting wave by a quadrature rule.

A further advantage of our method is that oblique shocks occurring in the solution of (4.2) (resp. (4.3)) can be accurately resolved in the (ξ, η) region. This is in contrast to finite difference methods based on dimensional splitting for the corresponding problem in the (x, y) plane. Crandall [8] shows that error due to dimensional splitting in computing oblique shocks by any finite difference method can be significant.

Remarks. Possible applications of the GSC method to hyperbolic systems of the form

$$u_t + [f(u)]_x = 0, \qquad u, f \in \mathbb{R}^n, \ x \in \mathbb{R}, \ t > 0 \tag{4.4}$$

have not yet been investigated. An extension of our method is complicated by the fact that the components of the state vector u are coupled by the Jacobian $A(u) := df/du$. In the purely linear case when A is a constant matrix, system (4.4) can be diagonalized by a similarly transformation resulting in n linear decoupled transport equations. They can be easily solved, just by transport. Another special case is given by the system

$$u_{i,t} + [f_i(u_1, ..., u_i)]_x = 0, \qquad i = 1, ..., n,$$

which appears in the field of oil recovery, see [10]. Now, A is a lower triangular matrix and a solution can be constructed successively, since the solution of the kth equation depends only on the $(k-1)$ first states.

5. NUMERICAL EXAMPLES

In this section we show the results of some test problems for

$$u_t + [f(u)]_x = h(u), \qquad u(x, 0) = u_0(x) \tag{5.1}$$

computed by the GSC method. The calculations were performed on an IBM compatible AT 80386. In all examples the CPU time is between 0.54 and 3.19 s.

EXAMPLE 5.1.

$$f(u) = -\cos\left(\frac{\pi}{2}u\right)\cos\left(\frac{3\pi}{2}u\right), \qquad h \equiv 0, \ u_0(x) = \begin{cases} 1; \ x < 0 \\ -1; \ x > 0. \end{cases}$$

The flux function f has two inflection points in $[-1, 1]$. The uncorrected version given by the transport at time $t = 0.15$ together with the three shocks are shown in Fig. 5. The shocks are represented by the heavy lines. CPU time is 3.19 s.

EXAMPLE 5.2.

$$f(u) = \frac{u^3}{3}, \qquad h(u) = -u^2, \qquad u_0(x) = \begin{cases} -1; \ x < 0 \\ 2; \ x > 0. \end{cases}$$

The solution exists only for $t < 1$, since at $t = 1$ the solution escapes from every compact subset of the state space. Figure 6 shows the uncorrected version and the solution at $t = 0.4$, where the shock position is $x_S = 0.198$ and the CPU time is 0.54 s. At $t = 0.99$ the solution consists of a single shock with position $x_S = 48.72$, shown in Fig. 7; the CPU time is 1.14 s. The use of Godunov's scheme, for example,
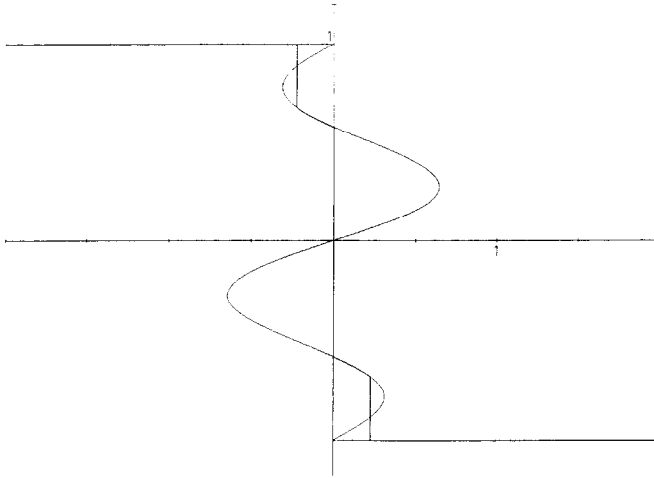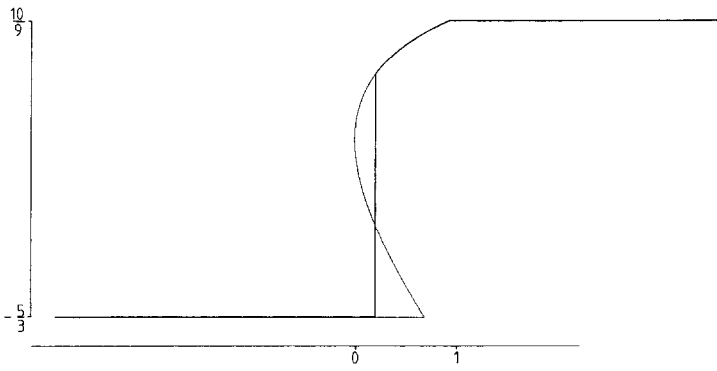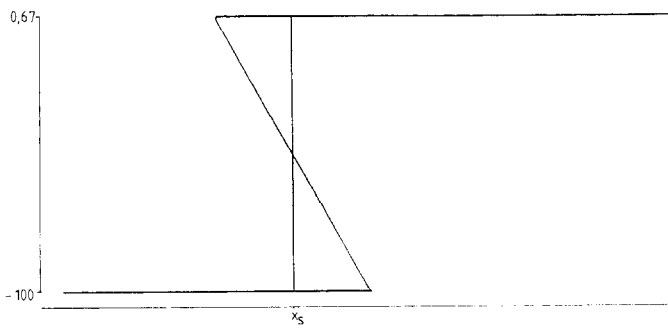
FIGURE 5

FIGURE 6

FIGURE 7

and Sod's operator splitting requires a small CFL number and results for $t \geq 0.75$ in numerical instabilities.

The next example also shows that more complicated problems can be solved effectively with the GSC method.

EXAMPLE 5.3.

$$f(u) = \frac{u^3}{3}, \qquad h(u) \equiv 0, \qquad u_0(x) = \begin{cases} 3 + \cos(x); \ x < 0 \\ -3 - \sin(x); \ x > 0. \end{cases}$$

In the x-interval $[-15, 16]$ the uncorrected version given by the transport at $t = 0.4$ together with the five shocks, represented by the heavy lines, are shown in Fig. 8. The shock posititions are -7.38; -1.097; 1.091; 6.9; 13.18 and the CPU time is 2.20 s.

EXAMPLE 5.4 (Buckley–Leverett equation). The simultaneous one-dimensional flow of two immiscible fluids through a porous medium in the absence of capillary pressure and gravitational forces can be described by the equation given by Buckley and Leverett (BL) in [4]. We consider the flow of oil and water through sand and denote by u the water saturation in the sand. Then the BL equation is

$$u_t + \frac{Q}{\phi} [f(u)]_x = 0,$$

where $f(u) = [(1 + \alpha k_o(u))/k_w(u)]^{-1}$ is the flux function of the flow stream, k_o and k_w are the relative permeabilities of sand to oil and to water, respectively; $\alpha = \mu_w/\mu_o$
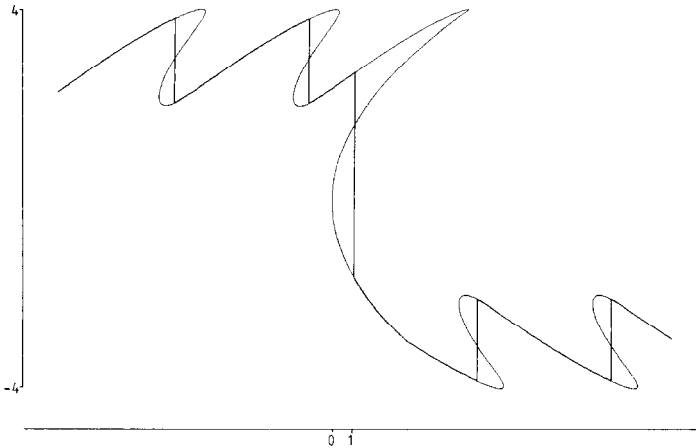

FIGURE 8

is a constant, μ_w (resp. μ_o) are the viscosities of water (resp. oil). Q is the total flow and ϕ is the porosity.

We remark that f has the following properties: df/du is nonnegative and has exactly one interior maximum and one inflection point in $[0, 1]$.

We report here on the results of the GSC method with the model flux

$$f(u) = \frac{u^2}{u^2 + \alpha(1-u)^2}. \tag{5.3}$$

The considered domain is $0 \leqslant u \leqslant 1$ and we take $\alpha = \frac{1}{2}$, $Q/\phi = 1$. The (continuous) initial condition is

$$u_0(x) = \begin{cases} 1; & x < 0 \\ \dfrac{0.1}{x + 0.1}; & 0 < x < 1 \\ \dfrac{1}{11}; & x > 1. \end{cases}$$

In Fig. 9 we show the solution at time 0.2 and in Fig. 10 at time 1. The CPU time at $t = 1$ is 2.68 s, which is smaller than the CPU time 2.92 at $t = 0.2$! The reason for this phenomena is that more iterations for comparing smaller areas may be needed, because shock positions are computed with accuracy 10^{-2}. We quote from [7] that the finite difference methods employ exceedingly small time steps, resulting in excessive computational requirements.
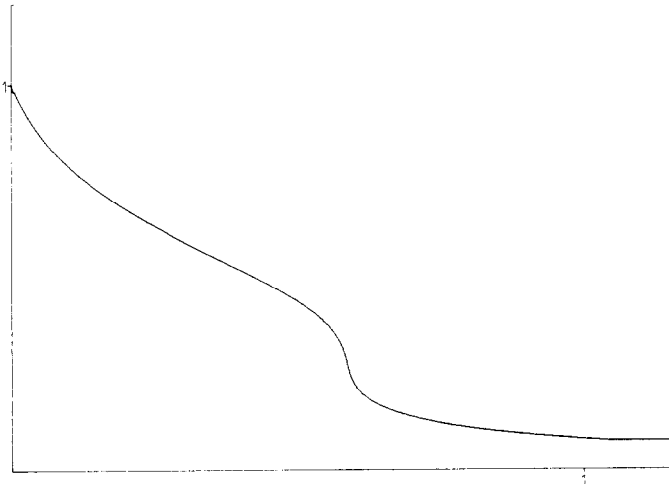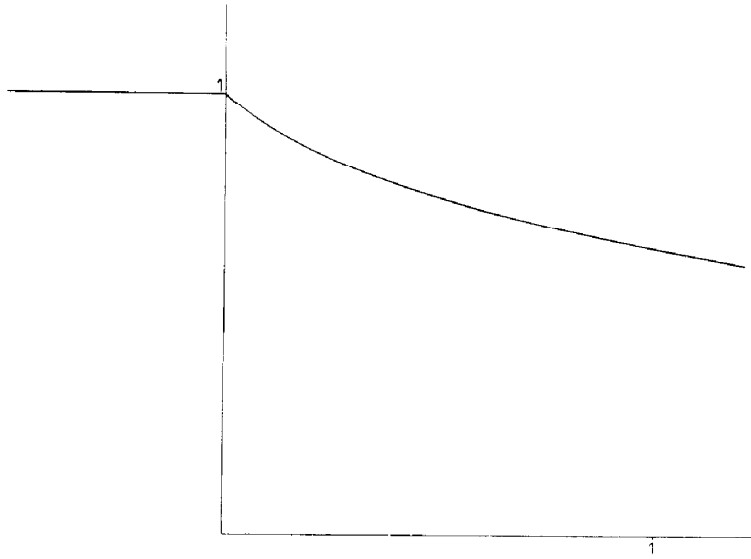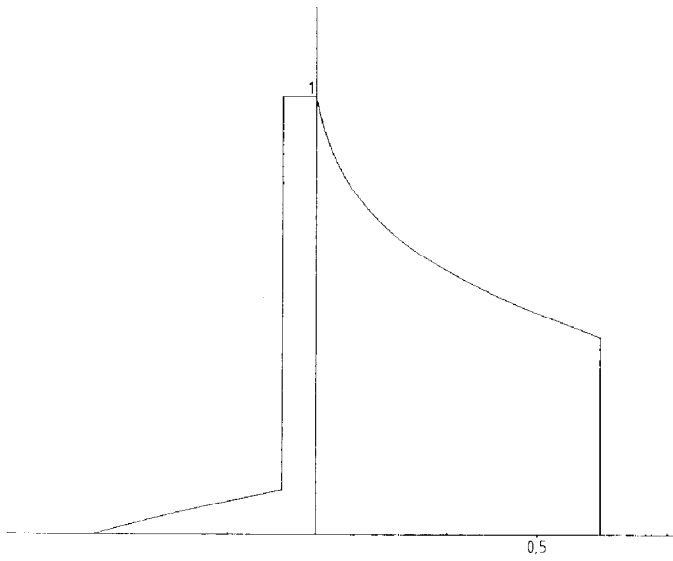

FIGURE 9

FIGURE 10

FIGURE 11

The final test problem (5.5) is taken from [6] and resolves the ambiguities of the example in the Introduction. The flux is given in (5.3) with $\alpha = 0.25$ and

$$u_0(x) = \begin{cases} 0, & x < -\frac{1}{2} \\ 1, & -\frac{1}{2} < x < 0 \\ 0, & x > 0. \end{cases}$$

The solution is shown at time 0.4; CPU time is 3.13 s.

SUMMARY

A complete class of nonlinear partial differential equations, namely the Cauchy problem (1.1) for generic cases, has been solved effectively by the GSC method, no matter how complicated the flux and the initial condition is. All test problems clearly demonstrate the superiority of our method in quality and computing time compared to any other numerical method. Furthermore, GSC is the fastest method known to the authors solving the considered equation for larger times.

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